RESEARCH NOTE

Eliminating the Fixed Predicates from a Circumscription

Johan de Kleer
Xerox Palo Alto Research Center, 3333 Coyote Hill Road, Palo Alto, CA 94304, U.S.A.

Kurt Konolige*
Artificial Intelligence Center, SRI International, 333 Ravenswood, Menlo Park, CA 94025, U.S.A.

ABSTRACT

Parallel predicate circumscription is the primary circumscriptive technique used in formalizing commonsense reasoning. In this paper we present a direct syntactic construction for transforming any parallel predicate circumscription using fixed predicates into an equivalent one which does not. Thus, we show that predicate circumscription is no more expressive with fixed predicates than without. We extend this result to prioritized circumscription. These results are expected to be useful for comparing circumscription to other nonmonotonic formalisms (such as autoepistemic logic and assumption-based truth maintenance) and for implementing fixed predicates.

Circumscription is one of the most well-developed formalizations of nonmonotonic reasoning. It exists in many variants, including domain circumscription [10], predicate circumscription [9], parallel predicate circumscription [9], formula circumscription [7], prioritized circumscription [4, 7], and pointwise circumscription [6]. All of these except the first and last are generalized by parallel predicate circumscription, which is currently the primary circumscriptive technique used in formalizing commonsense reasoning (see, for example, [5, 7]).

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In its most general form, parallel predicate circumscription (hereafter, the term "parallel" is understood) is a second-order formula that is true exactly in the P/Q-minimal models of a (first-order) sentence A, where P and Q are tuples of predicate constants appearing in A. A model m of A is P/Q-minimal if the extension of P in m is a subset of its extension in every other model of A with the same universe and extension of Q. The predicates Q are called *fixed* because their extensions must be equal if two models are to be comparable; all other predicate constants of A not in P or Q are called *varying* because their extensions are ignored in determining minimal models.

Fixed predicates, while useful from a representational point of view, complicate the analysis of predicate circumscription. In this note we present a direct syntactic construction for transforming any predicate circumscription using fixed predicates to one which does not. Thus predicate circumscription is no more expressive with fixed predicates than without. We further extend this result to prioritized circumscription, which is reducible to a conjunction of predicate circumscriptions. Finally, we show that a similar transformation exists for the first-order schema version of predicate circumscription, which is incomplete with respect to the minimal-model semantics. We expect these results to be useful both from an analytic point of view, i.e., comparing circumscription to other nonmonotonic formalisms such as autoepistemic logic [11] and assumption-based truth maintenance systems [3], and by extension, indicating a method for implementing fixed predicates in those formalisms which can be thought of as minimizing the extensions of predicates.

As may be inferred from the above discussion, we are considering circumscription from a semantic point of view—it selects certain models based on a minimization criterion. It is in this sense that parallel predicate circumscription generalizes many other forms of circumscription, because it allows a more general specification of minimization. This minimization criterion can be expressed using the second-order sentence

$$A(P, Z) \wedge \neg\exists pz(A(p, z) \wedge p < P)\,,$$

where $A(P, Z)$ is a first-order sentence with tuples of predicate constants P and Z, and $p < P$ is a formula expressing the fact that the extension of each member of p is a subset of the corresponding member of P, and one of them is a proper subset. The above sentence, which we abbreviate Circum(A; P; Z), represents the circumscription of sentence A, where the predicates P are minimized, the predicates Z vary, and the remaining predicates Q of A are fixed. The models of Circum(A; P; Z) are exactly the P/Q-minimal models of A (see [4, Proposition 1]).[1]

[1] The proof of this proposition does not appear in the cited publication, and has not yet been published.

Our main result, which we now present, is that Circum(A; P; Z) is "equivalent" to a circumscription with a slightly different A with no fixed predicates. As the construction we use extends the vocabulary of A, we must take this into account when defining the concept of equivalence. The requisite notion is that of *conservative extension*. Let L be the language of a second-order sentence S; we say that a sentence S' is a conservative extension of S just in case every sentence of L is true in all models of S exactly when it is true in all models of S'.[2]

We need the following definitions: Call two predicates with the same arity *similar*. If q and q' are similar predicates, we abbreviate $\forall x.q(x) \equiv \neg q'(x)$ by $q = \neg q'$.

Theorem 1. *Let q be a predicate of A not in P or Z, and let q' be a predicate not in A. Then*

$$\text{Circum}(A \wedge (q = \neg q'); P, q, q'; Z)$$

is a conservative extension of Circum(A; P; Z).

Proof. We exploit the minimal-model semantics of Circum. Let A' be the sentence $A \wedge (q = \neg q')$, and Q' the tuple Q with q removed. The models of Circum(A; P; Z) are the P/Q-minimal models of A, and the models of Circum($A \wedge (q = \neg q')$; P, q, q'; Z) are the $P, q, q'/Q'$-minimal models of A'. We will show that every sentence using the vocabulary of A is true in a model of Circum(A; P; Z) if and only if it is true in a model of Circum($A \wedge (q = \neg q')$; P, q, q'; Z).

Let m be a model of A; by the *expansion* of m we mean the interpretation m' that is the same as m, but with the addition of a relation for q' which is the complement of the relation for q. It is easy to see that m' is a model of A'. Further, any sentence using the vocabulary of A that is true in m is also true in m' (see [1, Exercise 1.3.3]).

Let m' be a model of A'. By the *reduct* of m' we mean the interpretation m that is the same as m', but without the relation for q'. Obviously, m is a model of A, and further, every sentence using the vocabulary of A that is true in m' is also true in m. Reduction is the full inverse of expansion, that is, the reduct of the expansion of m is m, and the expansion of the reduct of m is m. To see that the latter half of this statement is true, observe that in *every* model of A', the extension of q' is the complement of the extension of q.

[2] For first-order theories, the concept of conservative extension is usually defined in an equivalent proof-theoretic manner by using "provable from S" in place of "true in all models of S." For second-order languages there is no satisfactory proof theory, and the model-theoretic notions must be used.

Given the above remarks, it suffices to show that the expansion of every P/Q-minimal model of A is $P, q, q'/Q'$-minimal, and conversely, that the reduct of every $P, q, q'/Q'$-minimal model of A' is P/Q-minimal.

(1) Let m be a P/Q-minimal model of A, and let m' be its expansion. Suppose m' is not $P, q, q'/Q'$-minimal, so there is another model m_1' of A' with the same universe which agrees with m' on Q', but is less than m' on the predicates P, q, q'. This implies that, for all predicates e in P, q, q', we must have:

$$[\![e]\!]_{m_1'} \subseteq [\![e]\!]_{m'} .$$

In any model of A', the interpretation of q is the complement of the interpretation of q'. This is compatible with the above constraint only when

$$[\![q]\!]_{m_1'} = [\![q]\!]_{m'} , \qquad [\![q']\!]_{m_1'} = [\![q']\!]_{m'} .$$

Thus, m_1' must be less than m' on the extension of the predicates P, so for some predicate p of P, we have

$$[\![p]\!]_{m_1'} \subset [\![p]\!]_{m'} .$$

Let the reduct of m_1' be m_1; the reduct of m' is m. All of the above relations on extensions also hold for the reducts (except for those involving q'); therefore m cannot be P/Q-minimal, because m_1 agrees with it on the extensions of Q, and is less than it on the extensions of P. Hence m' must be $P, q, q'/Q'$-minimal.

(2) Let m' be a $P, q, q'/Q'$-minimal model of A', and let m be its reduct. Suppose m is not P/Q-minimal, so there is another model m_1 of A with the same universe which agrees with m on Q, and is less than m on P. The expansion m_1' of m_1 agrees with the expansion m' of m on all Q and q', and is less than m' on P. This means that m' is not a $P, q, q'/Q'$-minimal model of A', a contradiction. □

Remarks. According to the theorem, a fixed predicate q can be transformed into a minimized predicate by simultaneously minimizing its negation q'. The varying predicates are unchanged. The reason this transformation works is that any attempt to minimize q will cause q' to be larger, so that no changes to the extension of q are made by the transformed circumscription.

The general formulation of predicate circumscription also allows functions to be varied or fixed. Obviously, it is impossible to account for fixed functions by the same method, since minimization cannot be applied to functions. Functions are, however, an inessential extension of the predicate calculus with equality, and can always be replaced by an equivalent formulation in terms of predicates. For example, the sentence $P(f(a))$ is equivalent to

$$\begin{aligned}&\forall x.\exists! y.F(x, y)\\&\quad\wedge \exists! zA(z)\\&\quad\wedge \forall xy.F(x, y) \wedge A(y) \wedge P(x)\,.\end{aligned}$$

If f or a is a fixed function, we can minimize the corresponding predicates F or A and their complements.

Curiously enough, it is also possible to eliminate the *variable* predicates and functions from a circumscription (see [4]). However, this transformation adds a second-order predicate variable to A, which is not always reducible to a first-order sentence. The elimination of fixed predicates involves only first-order additions to A.

By successive application of this theorem, any predicate circumscription with fixed predicates Q can be transformed into an equivalent circumscription with no fixed predicates. If Q is a tuple of predicates $q_1, \ldots, q_n$, and Q' is a tuple of similar predicates $q'_1, \ldots, q'_n$, then by $Q = \neg Q'$ we mean $\bigwedge_{i=1}^{n} \forall \boldsymbol{x}.q_i(\boldsymbol{x}) = \neg q'_i(\boldsymbol{x})$.

Corollary 2. *Let Q be the predicates of A not contained in P or Z, and let Q' be predicates not in A with arities of the corresponding predicates in Q. Then*

$$\text{Circum}(A \wedge (Q = \neg Q'); P, Q, Q'; Z)$$

is a conservative extension of Circum(A; P; Z).

We can extend these results to prioritized circumscription, because the latter can be written as a conjunction of predicate circumscriptions.

Theorem 3 (Lifschitz [4]). Circum(A; $P^1 > \cdots > P^k$; Z) *is equivalent to*

$$\bigwedge_{i=1}^{k} \text{Circum}(A; P^i, \ldots, P^k; Z)\,.$$

By applying Corollary 2 to the elements of this conjunction, the fixed predicates can be eliminated.

Theorem 4. *Let P^0 be the fixed predicates of A: those not in Z or any P^i for $i > 0$. Let P'^j be predicates similar to P^j, but not in A. Then*

$$\bigwedge_{i=1}^{k} \text{Circum}\Big(A \wedge \bigwedge_{j=0}^{i-1} P^j = \neg P'^j; P^0, P'^0, \ldots, P^k, P'^k; Z\Big)$$

is a conservative extension of Circ(A; $P^1 > \cdots > P^k$; Z).

As a corollary, any prioritized circumscription can be transformed into one in which there are no fixed predicates.

Corollary 5.

$$\mathrm{Circum}(A \wedge (P^0 = \neg P'^0); P^0, P'^0, P^1 > \cdots > P^k; Z)$$

is a conservative extension of Circum(A; $P^1 > \cdots > P^k$; Z).

Predicate circumscription was originally defined proof-theoretically (in [9] by means of a first-order schema in which no predicates were allowed to vary; subsequently the schema was generalized to include varying predicates [8]. It is known that this schema is sound but incomplete with respect to the minimal-model semantics: there are first-order sentences true in all minimal models of A which are not derivable from the circumscriptive schema for A [2]. The question then arises as to whether there is a transformation for the schema (similar to that for Circum) which eliminates the fixed predicates, but results in an equivalent schema. The answer is affirmative, as we now show.

Let $A(P, Z)$ be a first-order sentence with predicate tuples P and Z. By a *formula predicate* we mean an expression of the form $\lambda x F$, where F is a formula with the free variables x. Let P^+ and Z^+ be tuples of formula predicates with arities corresponding to the predicates of P and Z. Then the schema Circ(A; P; Z) is given by

$$A \wedge (A(P^+, Z^+) \wedge P^+ \leqslant P \supset P^+ = P) .$$

Theorem 6. *Let q be a predicate of A not in P or Z, and let q' be a predicate not in A. Then*

$$\mathrm{Circ}(A \wedge (q = \neg q'); P, q, q'; Z)$$

is a conservative extension of Circ(A; P; Z).

Proof. By definition, Circ($A \wedge (q = \neg q')$; P, q, q'; Z) is

$$\begin{aligned} & A \wedge q = \neg q' \\ & \quad \wedge (A(P^+, Z^+, q^+) \wedge q^+ = \neg q'^+ \wedge P^+ \leqslant P \wedge q^+ \leqslant q \wedge q'^+ \leqslant q' \\ & \qquad \supset P^+ = P \wedge q^+ = q \wedge q'^+ = q') . \end{aligned}$$

If we substitute q for q^+ and $\neg q'$ for $\neg q'^+$, then it is easy to see that the above expression reduces to Circ(A; P; Z). Thus every instance of Circ(A; P; Z) is an instance of Circ($A \wedge (q = \neg q')$; P, q, q'; Z).

In the converse direction, we must show that every sentence S not containing the symbol q' that is a consequence of Circ($A \wedge (q = \neg q')$; P, q, q'; Z) is also a consequence of Circ(A; P; Z). We can do this by showing that every model of Circ(A; P; Z) is a reduct of some model of Circ($A \wedge (q = \neg q')$; P, q, q'; Z). Since S is true in the reduct of any model of Circ($A \wedge (q = \neg q')$; P, q, q'; Z), this will establish that S is true in any model of Circ(A; P; Z).

Consider any model m of Circ(A; P; Z). Extend m to m' by adding an interpretation for q' such that $q = \neg q'$. Now m' obviously satisfies the first two conjuncts of the definition of Circ($A \wedge (q = \neg q')$; P, q, q'; Z) above. Let us rewrite the third conjunct in the following way:

$$(q^+ = \neg q'^+ \wedge q^+ \leqslant q \wedge q'^+ \leqslant q')$$
$$\supset (A(P^+, Z^+, q^+) \wedge P^+ \leqslant P \supset P^+ = P \wedge q^+ = q \wedge q'^+ = q') .$$

If the antecedent to the first implication is false in m', then m' is a model of Circ($A \wedge (q = \neg q')$; P, q, q'; Z). So assume that the antecedent to the first implication is true in m'. By noting that $q = \neg q'$ and $q^+ = \neg q'^+$, we can convert $q'^+ \leqslant q'$ into the equivalent form $q^+ \geqslant q$ by simple first-order manipulations. Hence $q^+ = q$, and by a similar argument $q'^+ = q'$. Thus, in this case the whole expression is equivalent to

$$A(P^+, Z^+, q) \wedge P^+ \leqslant P \supset P^+ = P ,$$

which is satisfied by m (and by extension, m'). Thus m' is a model of Circ($A \wedge (q = \neg q')$; P, q, q'; Z). □

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